Presence of exotic matter in the Cooperstock and Tieu galaxy model

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Abstract

We analyze the presence of an additional singular thin disk in the recent General Relativistic model of galactic gravitational field proposed by Cooperstock and Tieu. The physical variables of the disk's energy-momentum tensor are calculated. We show that the disk is made of exotic matter, either cosmic strings or struts with negative energy density.

1 Introduction

Recently Cooperstock and Tieu [1] have proposed a General Relativistic galactic model, which is essentially composed of a uniformly rotating fluid without pressure, and fitted flat rotation curves without the (apparent) need of dark matter with unusual properties. However, Korzyński [2] argued that their model has an additional source of gravitational field in the form of a rotating flat disk on the galactic plane, and thus should be considered "unphysical". We think that the fact of having a flat disk is not a relevant problem if the matter content of the disk is made of usual matter. We agree with Korzyński that some of the equations derived by Cooperstock and Tieu are not satisfied on the disk.

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We shall compute the matter content of an additional thin rotating disk in Cooperstock and Tieu's model, and show that the disk is composed of exotic matter.

2 Cooperstock and Tieu galaxy model

In [1] the authors use the van Stockum metric [3]

$$\mathrm{d}s^2 = (\mathrm{d}t - N\mathrm{d}\varphi)^2 - r^2\mathrm{d}\varphi^2 - e^{\nu}(\mathrm{d}r^2 + \mathrm{d}z^2), \tag{1}$$

and propose a solution of the following form to fit the galactic rotation curves:

$$\Phi = \sum_n C_n e^{-k_n|z|} J_0(k_n r), \tag{2}$$

$$N = r\frac{\partial \Phi}{\partial r}. \tag{3}$$

The absolute value of z is introduced to provide reflection symmetry of the matter distribution with respect to the plane $z = 0$. Although the presence of $|z|$ does not modify the field equation for the density distribution

$$\rho = \frac{c^2}{8\pi G}\frac{N_{,r}^2 + N_{,z}^2}{r^2}, \tag{4}$$

it introduces a distributional singularity at $z = 0$ in the equation for the metric function N:

$$N_{,rr} + N_{,zz} - \frac{N_{,r}}{r} = 0, \tag{5}$$

and an additional distributional term in the energy-momentum tensor T_{ab} [2]. We shall now calculate explicitly all the distributional components of T_{ab}.

3 The van Stockum class of metrics

For metric equation (1), the exact Einstein field equations can be cast as [3], [4]

$$N_{,rr} + N_{,zz} - \frac{N_{,r}}{r} = 0, \tag{6a}$$

$$\nu_{,z} = -\frac{N_{,r}N_{,z}}{r}, \qquad \nu_{,r} = \frac{N_{,z}^2 - N_{,r}^2}{2r}, \tag{6b}$$

$$\rho = \frac{1}{r^2 e^{\nu}}\left(N_{,r}^2 + N_{,z}^2\right), \tag{6c}$$

(we use units such that $c = 8\pi G = 1$). The introduction of an absolute value of z in a solution of Eq. (6a) can be thought as doing a transformation $z \to |z|$ in the metric functions N and ν (every transformation applied on N will also affect the function ν by Eq. (6b)). In the Einstein tensor we have first and second derivatives of z. Since $\partial_z |z| = 2\vartheta(z) - 1$ and $\partial_{zz}|z| = 2\delta(z)$, where $\vartheta(z)$ and $\delta(z)$ are, respectively, the Heaviside function and the Dirac distribution, the Einstein equations yield an distributional energy-momentum tensor $T_{ab} = Q_{ab}\delta(z)$, with [5]

$$Q^a_b = \frac{1}{2}\left[b^{az}\delta^z_b - b^{zz}\delta^a_b + g^{az}b^z_b - g^{zz}b^a_b + b^c_c(g^{zz}\delta^a_b - g^{az}\delta^z_b)\right]. \tag{7}$$

Here b_{ab} denote the discontinuity functions of the first derivatives with respect of z of the metric tensor on the plane $z = 0$:

$$b_{ab} = g_{ab,z}\big|_{z=0^+} - g_{ab,z}\big|_{z=0^-} = 2\, g_{ab,z}\big|_{z=0^+}. \tag{8}$$

Using Eq. (7)–(8) and metric (1), the only non-zero components of Q^a_b are

$$Q^t_t = \frac{1}{e^\nu}\left(\frac{NN_{,z}}{r^2} - \nu_{,z}\right), \tag{9a}$$

$$Q^t_\varphi = -\frac{N_{,z}}{e^\nu}\left(1 + \frac{N^2}{r^2}\right), \tag{9b}$$

$$Q^\varphi_t = \frac{N_{,z}}{r^2 e^\nu}, \tag{9c}$$

$$Q^\varphi_\varphi = -\frac{1}{e^\nu}\left(\frac{NN_{,z}}{r^2} + \nu_{,z}\right). \tag{9d}$$

The physical variables of the disk are obtained by solving the eigenvalue problem for Q^a_b: $Q^a_b\xi^b = \lambda\xi^a$, and has the solutions

$$\lambda_\pm = \frac{T}{2} \pm \frac{\sqrt{D}}{2}, \quad \text{where} \tag{10}$$

$$T = Q^t_t + Q^\varphi_\varphi, \qquad D = (Q^t_t - Q^\varphi_\varphi)^2 + 4Q^t_\varphi Q^\varphi_t. \tag{11}$$

Using Eq. (9a)–(9d), we obtain

$$T = -\frac{2\nu_{,z}}{e^\nu}, \qquad D = -\frac{4N^2_{,z}}{r^2 e^{2\nu}}. \tag{12}$$

The discriminant $D < 0$ characterizes heat flow in the tangential direction [5], [6]. In this case, the surface energy density and azimuthal stresses are,

respectively, $\sigma = T/2$ and $P_\varphi = -T/2$. Thus, the disk is composed of matter with an equation of state $P_\varphi = -\sigma$. If $\sigma > 0$ we have tensions and it may be interpreted as an equation of state of matter formed by concentric loops of cosmic strings [7]. If $\sigma < 0$ we also have exotic matter with negative energy density. Objects with this equation of state are known in the literature as struts and they appear to stabilize certain superpositions of static isolated bodies in General Relativity (see, for instance, [8]). We also note that all the above results are exact.

Although the proposed galactic model does not really resolves galactic rotation without the presence of exotic matter, we believe that the idea of treating the non-linear galactic dynamical problem in the context of General Relativity is quite interesting and should be further investigated, specially the rotating models where we have the non-Newtonian effect of dragging of inertial frames; a modest step in this direction is presented in [9].

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